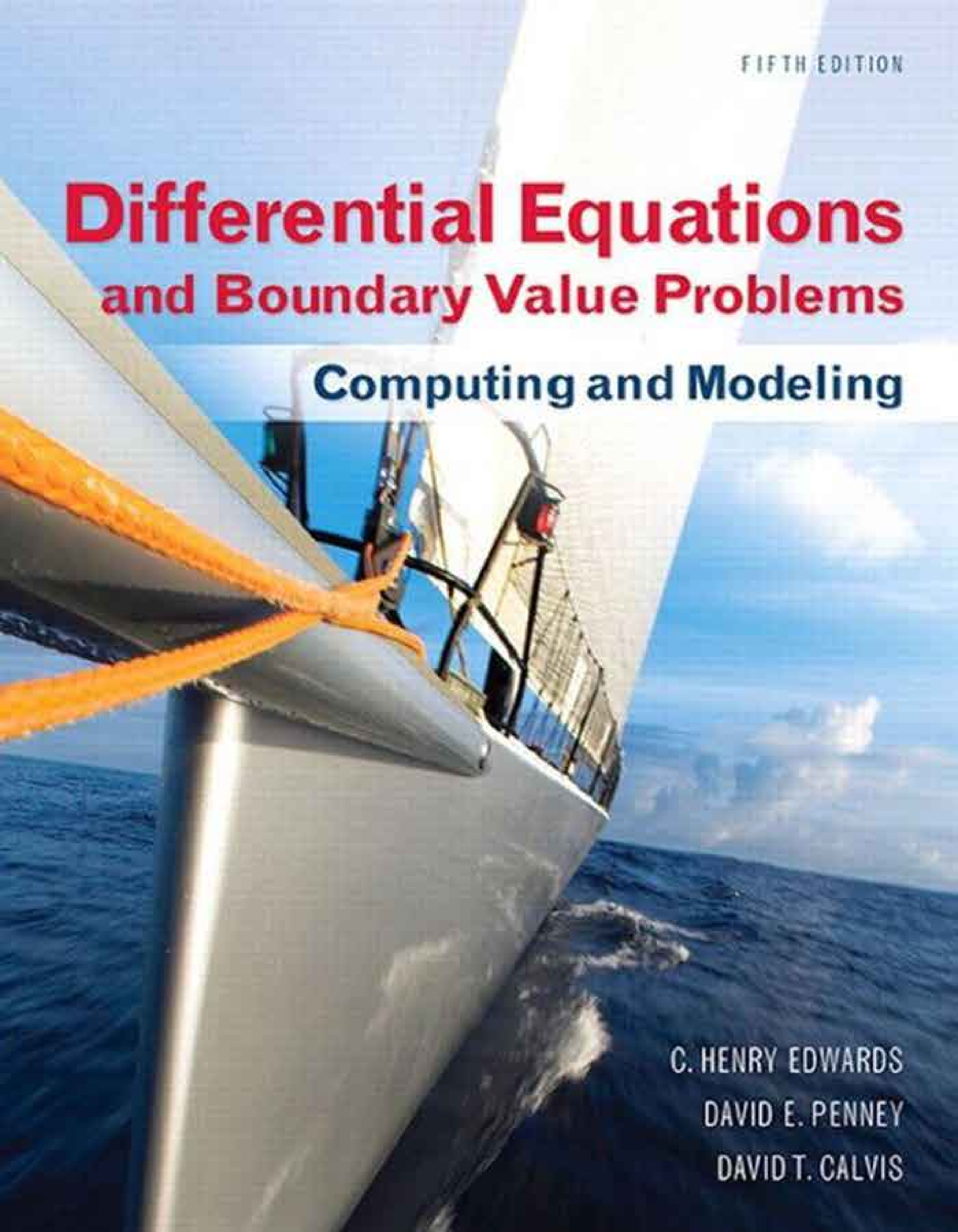

FIFTH EDITION

Differential Equations and Boundary Value Problems

Computing and Modeling

C. HENRY EDWARDS

DAVID E. PENNEY

DAVID T. CALVIS

DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

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with the assistance of

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CONTENTS

APPLICATION MODULES

The modules listed here follow the indicated sections in the text. Most provide computing projects that illustrate the content of the corresponding text sections. *Maple*, *Mathematica*, and MATLAB versions of these investigations are included in the Applications Manual that accompanies this text.

PREFACE

This is a textbook for the standard introductory differential equations course taken by science and engineering students. Its updated content reflects the wide availability of technical computing environments like *Maple*, *Mathematica*, and MATLAB that now are used extensively by practicing engineers and scientists. The traditional manual and symbolic methods are augmented with coverage also of qualitative and computer-based methods that employ numerical computation and graphical visualization to develop greater conceptual understanding. A bonus of this more comprehensive approach is accessibility to a wider range of more realistic applications of differential equations.

Principal Features of This Revision

This 5th edition is a comprehensive and wide-ranging revision.

In addition to fine-tuning the exposition (both text and graphics) in numerous sections throughout the book, new applications have been inserted (including biological), and we have exploited throughout the new interactive computer technology that is now available to students on devices ranging from desktop and laptop computers to smart phones and graphing calculators. It also utilizes computer algebra systems such as *Mathematica*, Maple, and MATLAB as well as online web sites such as Wolfram|Alpha.

However, with a single exception of a new section inserted in Chapter 5 (noted below), the classtested table of contents of the book remains unchanged. Therefore, instructors' notes and syllabi will not require revision to continue teaching with this new edition.

A conspicuous feature of this edition is the insertion of about 80 new computer-generated figures, many of them illustrating how interactive computer applications with slider bars or touchpad controls can be used to change initial values or parameters in a differential equation, allowing the user to immediately see in real time the resulting changes in the structure of its solutions.

Some illustrations of the various types of revision and updating exhibited in this edition:

New Interactive Technology and Graphics New figures inserted throughout illustrate the facility offered by modern computing technology platforms for the user to interactively vary initial conditions and other parameters in real time. Thus, using a mouse or touchpad, the initial point for an initial value problem can be dragged to a new location, and the corresponding solution curve is automatically redrawn and dragged along with its initial point. For instance, see the Sections 1.3 (page 28) application module and 3.1 (page 148). Using slider bars in an interactive graphic, the coefficients or other parameters in a linear system can be varied, and the corresponding changes in its direction field and phase plane portrait are automatically shown; for instance, see the application module for Section 5.3 (page 319). The number of terms used from an infinite series solution of a differential equation can be varied, and the resulting graphical change in the corresponding approximate solution is shown immediately; see the Section 8.2 application module (page 516).

New Exposition In a number of sections, new text and graphics have been inserted to enhance student understanding of the subject matter. For instance, see the treatments of separable equations in Section 1.4 (page 30), linear equations in Section 1.5 (page 45), isolated critical points in Sections 6.1 (page 372) and 6.2 (page 383), and the new example in Section 9.6 (page 618) showing a vibrating string with a momentary "flat spot." Examples and accompanying graphics have been updated in Sections 2.4–2.6, 4.2, and 4.3 to illustrate new graphing calculators.

New Content The single entirely new section for this edition is Section 5.3, which is devoted to the construction of a "gallery" of phase plane portraits illustrating all the possible geometric behaviors of solutions of the 2-dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. In motivation and preparation for the detailed study of eigenvalue-eigenvector methods in subsequent sections of Chapter 5 (which then follow in the same order as in the previous edition), Section 5.3 shows how the particular arrangements of eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$ correspond to identifiable patterns—"fingerprints," so to speak—in the phase plane portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. The resulting gallery is shown in the two pages of phase plane portraits that comprise Figure 5.3.16 (pages 315-316) at the end of the section. The new 5.3 application module (on dynamic phase plane portraits, page 319) shows how students can use interactive computer systems to "bring to life" this gallery, by allowing initial conditions, eigenvalues, and even eigenvectors to vary in real time. This dynamic approach is then illustrated with several new graphics inserted in the remainder of Chapter 5. Finally, for a new biological application, see the application module for Section 6.4, which now includes a substantial investigation (page 423) of the nonlinear FitzHugh-Nagumo equations in neuroscience, which were introduced to model the behavior of neurons in the nervous system.

Computing Features

The following features highlight the computing technology that distinguishes much of our exposition.

- Over 750 *computer-generated figures* show students vivid pictures of direction fields, solution curves, and phase plane portraits that bring symbolic solutions of differential equations to life.
- About 45 *application modules* follow key sections throughout the text. Most of these applications outline "technology neutral" investigations illustrating the use of technical computing systems and seek to actively engage students in the application of new technology.
- A fresh *numerical emphasis* that is afforded by the early introduction of numerical solution techniques in Chapter 2 (on mathematical models and numerical methods). Here and in Chapter 4, where numerical techniques for systems are treated, a concrete and tangible flavor is achieved by the inclusion of numerical algorithms presented in parallel fashion for systems ranging from graphing calculators to MATLAB.

Modeling Features

Mathematical modeling is a goal and constant motivation for the study of differential equations. To sample the range of applications in this text, take a look at the following questions:

- What explains the commonly observed time lag between indoor and outdoor daily temperature oscillations? (Section 1.5)
- What makes the difference between doomsday and extinction in alligator populations? (Section 2.1)
- How do a unicycle and a twoaxle car react differently to road bumps? (Sections 3.7 and 5.4)
- How can you predict the time of next perihelion passage of a newly observed comet? (Section 4.3)
- Why might an earthquake demolish one building and leave standing the one next door? (Section 5.4)
- What determines whether two species will live harmoniously together, or whether competition will result in the extinction of one of them and the survival of the other? (Section 6.3)
- Why and when does non-linearity lead to chaos in biological and mechanical systems? (Section 6.5)
- If a mass on a spring is periodically struck with a hammer, how does the behavior of the mass depend on the frequency of the hammer blows? (Section 7.6)
- Why are flagpoles hollow instead of solid? (Section 8.6)
- What explains the difference in the sounds of a guitar, a xylophone, and drum? (Sections 9.6, 10.2, and 10.4)

Organization and Content

We have reshaped the usual approach and sequence of topics to accommodate new technology and new perspectives. For instance:

- After a precis of first-order equations in Chapter 1 (though with the coverage of certain traditional symbolic methods streamlined a bit), Chapter 2 offers an early introduction to mathematical modeling, stability and qualitative properties of differential equations, and numerical methods—a combination of topics that frequently are dispersed later in an introductory course. Chapter 3 includes the standard methods of solution of linear differential equations of higher order, particularly those with constant coefficients, and provides an especially wide range of applications involving simple mechanical systems and electrical circuits; the chapter ends with an elementary treatment of endpoint problems and eigenvalues.
- Chapters 4 and 5 provide a flexible treatment of linear systems. Motivated by current trends in science and engineering education and practice, Chapter 4 offers an early, intuitive introduction to first-order systems, models, and numerical approximation techniques. Chapter 5 begins with a self-contained

treatment of the linear algebra that is needed, and then presents the eigenvalue approach to linear systems. It includes a wide range of applications (ranging from railway cars to earthquakes) of all the various cases of the eigenvalue method. Section 5.5 includes a fairly extensive treatment of matrix exponentials, which are exploited in Section 5.6 on nonhomogeneous linear systems.

- Chapter 6 on nonlinear systems and phenomena ranges from phase plane analysis to ecological and mechanical systems to a concluding section on chaos and bifurcation in dynamical systems. Section 6.5 presents an elementary introduction to such contemporary topics as period-doubling in biological and mechanical systems, the pitchfork diagram, and the Lorenz strange attractor (all illustrated with vivid computer graphics).
- Laplace transform methods (Chapter 7) and power series methods (Chapter 8) follow the material on linear and nonlinear systems, but can be covered at any earlier point (after Chapter 3) the instructor desires.
- Chapters 9 and 10 treat the applications of Fourier series, separation of variables, and Sturm-Liouville theory to partial differential equations and boundary value problems. After the introduction of Fourier series, the three classical equations—the wave and heat equations and Laplace's equation—are discussed in the last three sections of Chapter 9. The eigenvalue methods of Chapter 10 are developed sufficiently to include some rather significant and realistic applications.

This book includes enough material appropriately arranged for different courses varying in length from one quarter to two semesters. The briefer version Differential Equations: Computing and Modeling (0-321-81625-0) ends with Chapter 7 on Laplace transform methods (and thus omits the material on power series methods, Fourier series, separation of variables and partial differential equations).

Student and Instructor Resources

The answer section has been expanded considerably to increase its value as a learning aid. It now includes the answers to most odd-numbered problems plus a good many even-numbered ones. The **Instructor's Solutions Manual** (0-321-79701-9) available at `www.pearsonhighered.com/irc` provides worked-out solutions for most of the problems in the book, and the **Student Solutions Manual** (0-321-79700-0) contains solutions for most of the odd-numbered problems. These manuals have been reworked extensively for this edition with improved explanations and more details inserted in the solutions of many problems.

The approximately 45 application modules in the text contain additional problem and project material designed largely to engage students in the exploration and application of computational technology. These investigations are expanded considerably in the **Applications Manual** (0-321-79704-3) that accompanies the text and supplements it with additional and sometimes more challenging investigations. Each section in this manual has parallel subsections **Using *Maple***, **Using *Mathematica***, and **Using MATLAB** that detail the applicable methods and techniques of each system, and will afford student users an opportunity to compare the merits and styles of different computational systems. These materials—as well as the text of the **Applications Manual** itself—are freely available at the web site `www.pearsonhighered.com/mathstatsresources`.

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1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

Example 3 Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

$$\frac{dT}{dt} = -k(T - A), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then $dT/dt < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $dT/dt > 0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

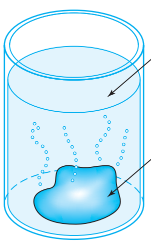

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Example 4 Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = -k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/A$ is a constant. ■

Example 5 The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

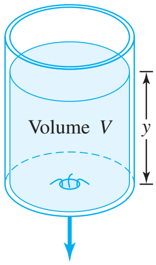

FIGURE 1.1.2. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \tag{7}$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6 Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$
$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$ ■

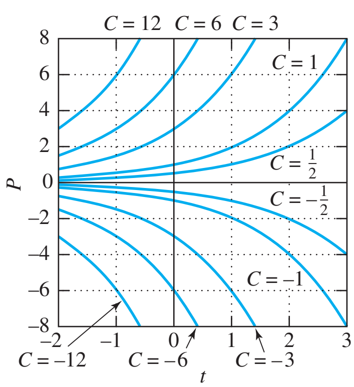

FIGURE 1.1.3. Graphs of $P(t) = Ce^{kt}$ with $k = \ln 2$.

The condition $P(0) = 1000$ in Example 6 is called an **initial condition** because we frequently write differential equations for which $t = 0$ is the "starting time." Figure 1.1.3 shows several different graphs of the form $P(t) = Ce^{kt}$ with $k = \ln 2$. The graphs of all the infinitely many solutions of $dP/dt = kP$ in fact fill the entire two-dimensional plane, and no two intersect. Moreover, the selection of any one point P_0 on the P-axis amounts to a determination of $P(0)$. Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ determines a unique solution agreeing with the given data.

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

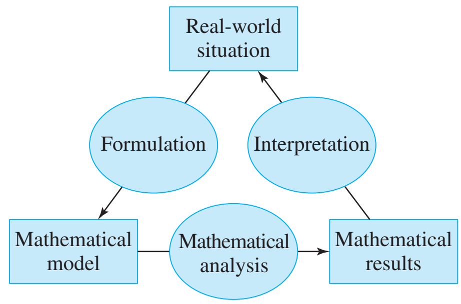

FIGURE 1.1.4. The process of mathematical modeling.

In the population example, the real-world problem is that of determining the population at some future time. A **mathematical model** consists of a list of variables (P and t) that describe the given situation, together with one or more equations relating these variables ($dP/dt = kP$, $P(0) = P_0$) that are known or are assumed to hold. The mathematical analysis consists of solving these equations (here, for P as a function of t). Finally, we apply these mathematical results to attempt to answer the original real-world question.

As an example of this process, think of first formulating the mathematical model consisting of the equations $dP/dt = kP$, $P(0) = 1000$, describing the bacteria population of Example 6. Then our mathematical analysis there consisted of solving for the solution function $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ as our mathematical result. For an interpretation in terms of our real-world situation—the actual bacteria population—we substituted $t = 1.5$ to obtain the predicted population of $P(1.5) \approx 2828$ bacteria after 1.5 hours. If, for instance, the bacteria population is growing under ideal conditions of unlimited space and food supply, our prediction may be quite accurate, in which case we conclude that the mathematical model is adequate for studying this particular population.

On the other hand, it may turn out that no solution of the selected differential equation accurately fits the actual population we're studying. For instance, for *no* choice of the constants C and k does the solution $P(t) = Ce^{kt}$ in Eq. (7) accurately describe the actual growth of the human population of the world over the past few centuries. We must conclude that the differential equation $dP/dt = kP$ is inadequate for modeling the world population—which in recent decades has "leveled off" as compared with the steeply climbing graphs in the upper half ($P > 0$) of Fig. 1.1.3. With sufficient insight, we might formulate a new mathematical model including a perhaps more complicated differential equation, one that takes into account such factors as a limited food supply and the effect of increased population on birth and death rates. With the formulation of this new mathematical model, we may attempt to traverse once again the diagram of Fig. 1.1.4 in a counterclockwise manner. If we can solve the new differential equation, we get new solution functions to compare with the real-world population. Indeed, a successful population analysis may require refining the mathematical model still further as it is repeatedly measured against real-world experience.

But in Example 6 we simply ignored any complicating factors that might affect our bacteria population. This made the mathematical analysis quite simple, perhaps unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. If the model is so detailed that it fully represents the physical situation, then the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may be so inaccurate as to be useless. Thus there is an inevitable tradeoff between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process. Ways must be found to simplify the model mathematically without sacrificing essential features of the real-world situation.

Mathematical models are discussed throughout this book. The remainder of this introductory section is devoted to simple examples and to standard terminology used in discussing differential equations and their solutions.

Examples and Terminology

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C-x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" C. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1-x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0 \tag{10}$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$. ■

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \tag{13}$$

where F is a specific real-valued function of $n + 2$ variables.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \dots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on I.

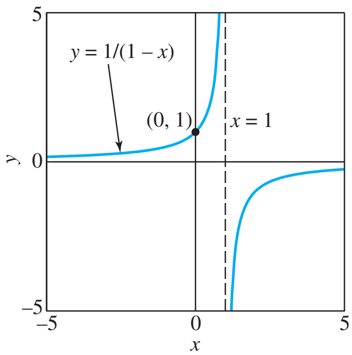

FIGURE 1.1.5. The solution of $y' = y^2$ defined by $y(x) = 1/(1 - x)$.

Remark Recall from elementary calculus that a differentiable function on an open interval is necessarily continuous there. This is why only a continuous function can qualify as a (differentiable) solution of a differential equation on an interval. ■

Example 7 *Continued*

Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1 - x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1 - x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$. ■

Example 9

If A and B are constants and

$$y(x) = A\cos 3x + B\sin 3x, \tag{14}$$

then two successive differentiations yield

$$y'(x) = -3A\sin 3x + 3B\cos 3x,$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

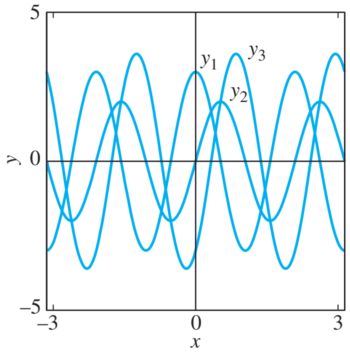

FIGURE 1.1.6. The three solutions $y_1(x) = 3\cos 3x$, $y_2(x) = 2\sin 3x$, and $y_3(x) = -3\cos 3x + 2\sin 3x$ of the differential equation $y'' + 9y = 0$.

Although the differential equations in (11) and (12) are exceptions to the general rule, we will see that an nth-order differential equation ordinarily has an n-parameter family of solutions—one involving n different arbitrary constants or parameters.

In both Eqs. (11) and (12), the appearance of y' as an implicitly defined function causes complications. For this reason, we will ordinarily assume that any differential equation under study can be solved explicitly for the highest derivative that appears; that is, that the equation can be written in the so-called *normal form*

$$y^{(n)} = G\left(x, y, y', y'', \dots, y^{(n-1)}\right), \tag{16}$$

where G is a real-valued function of $n+1$ variables. In addition, we will always seek only real-valued solutions unless we warn the reader otherwise.

All the differential equations we have mentioned so far are **ordinary** differential equations, meaning that the unknown function (dependent variable) depends on only a *single* independent variable. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a **partial** differential equation. For example, the temperature $u = u(x,t)$ of a long thin uniform rod at the point x at time t satisfies (under appropriate simple conditions) the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where k is a constant (called the *thermal diffusivity* of the rod). In Chapters 1 through 8 we will be concerned only with *ordinary* differential equations and will refer to them simply as differential equations.

In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing x_0.

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

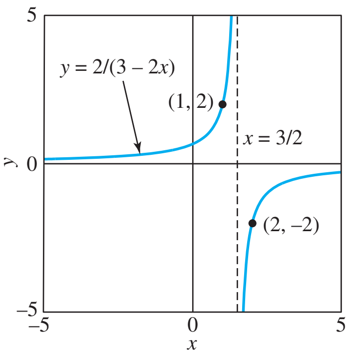

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3 - 2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3 - 2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$

38. $y' = y$

39. $xy' + y = 3x^2$

40. $(y')^2 + y^2 = 1$

41. $y' + y = e^x$

42. $y'' + y = 0$

Problems 43 through 46 concern the differential equation

$$\frac{dx}{dt} = kx^2,$$

where k is a constant.

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.
(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.
(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. Based on the result of Problem 43, how long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. Based on the result of Problem 43, long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$ m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

48. **(a)** Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

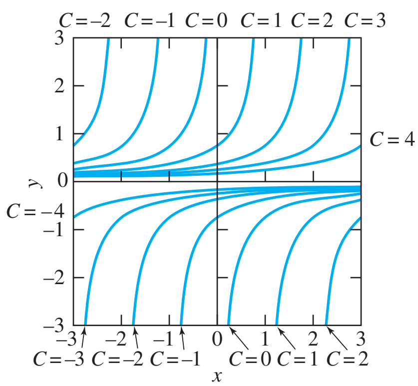

FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

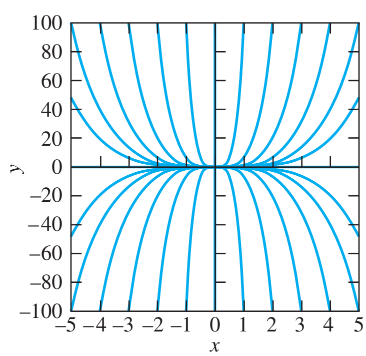

FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.